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A summation formula for Clausen's series ${}_3F_2(1)$ with an application to Goursat's function ${}_2F_2(x)$

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Abstract

A new summation formula for Clausen's series ${}_3F_2(1)$ is derived in two different ways and used to obtain a reduction formula for the Kampé de Fériet function $F^{p:2;0}_{q:2;0}[-x, x]$. The specialization $p = q = 0$ of the latter result reduces to a Kummer-type transformation formula for the generalized hypergeometric function ${}_2F_2(x)$ which has recently been deduced by R B Paris who employed other methods.

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1. Introduction

Recently, Paris [7] deduced a Kummer-type transformation formula for the generalized hypergeometric function ${}_2F_2(x)$, namely

$$ {}_2F_2\left(\begin{matrix} a, c+1; \\ b, c; \end{matrix}\ x\right) = \mathrm{e}^x\, {}_2F_2\left(\begin{matrix} b-a-1\alpha+1; \\ b, \alpha; \end{matrix}\ -x\right), \tag{1.1a} $$

where α is defined by

$$ \alpha \equiv \frac{c(1+a-b)}{a-c}. \tag{1.1b} $$

Equation (1.1a) is analogous to the well-known and much frequented Kummer's first transformation formula for the confluent hypergeometric function ${}_1F_1(x)$, which states that

$$ {}_1F_1\left(\begin{matrix} a; \\ b; \end{matrix}\ x\right) = \mathrm{e}^x\, {}_1F_1\left(\begin{matrix} b-a; \\ b; \end{matrix}\ -x\right). \tag{1.2} $$

The result given by equations (1.1) provides a generalization of a previous result (where $c = \frac{1}{2}a$ so that $\alpha = 1 + a - b$) due to Exton [3] which was rederived by Miller [6]. Moreover, the result given by equations (1.1) contains Kummer's equation (1.2) which can be seen by letting $b = c + 1$ in equations (1.1).

Paris deduced equations (1.1) in [7] by essentially employing the beta (integral) transform of the confluent function as a representation for ${}_2F_2(x)$, an addition theorem for the confluent function ${}_1F_1(x+y)$, and equation (1.2). In the present investigation, in section 2 we shall derive in two different ways an apparently new summation formula for Clausen's series ${}_3F_2(1)$ which we utilize in section 3 to obtain a reduction formula for the Kampé de Fériet (double generalized hypergeometric) function $F^{p:2;0}_{q:2;0}[-x,x]$. The specialization $p=q=0$ of the result just alluded to then immediately yields equations (1.1).

It is well known that generalized hypergeometric functions ${}_pF_q(x)$ appear ubiquitously as solutions to a plethora of problems in mathematics, statistics and mathematical physics. Thus the results given by equations (1.1) and (2.1) should eventually prove useful in a wide range of applications.

2. Summation formulae

We shall show for nonnegative integers n that

$$ {}_3F_2\left(\begin{matrix}-n, a, c+1; \\ b, c;\end{matrix}\ 1\right) = \frac{(b-a-1)_n(\alpha+1)_n}{(b)_n(\alpha)_n}, \tag{2.1}$$

where $(b)_n \equiv \Gamma(b+n)/\Gamma(b)$ is the Pochhammer symbol and α is given by equation (1.1*b*). This summation formula is curiously much different from others of its kind since $\alpha = c(1+a-b)/(a-c)$ is obviously not a linear first degree combination of the free (complex) parameters a, b and c; moreover, it is not even a rational quotient of the latter because of the presence of c in the numerator of α. (See [2, 9] for other summation theorems for ${}_3F_2(1)$.) We shall see that equation (2.1) is a corollary of the following lemma.

Lemma 1. *For* $\mathrm{Re}(b-a-f)>1$ *and* α *defined by equation (1.1b)*

$$ {}_3F_2\left(\begin{matrix}f, a, c+1; \\ b, c;\end{matrix}\ 1\right) = \frac{(c-a)(\alpha-f)}{c}\frac{\Gamma(b)\Gamma(b-a-f-1)}{\Gamma(b-a)\Gamma(b-f)}. \tag{2.2}$$

To show this we observe that $(c+1)_p/(c)_p = 1+p/c$ and so we have

$$ {}_3F_2\left(\begin{matrix}f, a, c+1; \\ b, c;\end{matrix}\ 1\right) = \sum_{p=0}^{\infty}\frac{(f)_p(a)_p}{(b)_p p!} + \frac{1}{c}\sum_{p=1}^{\infty}\frac{(f)_p(a)_p}{(b)_p(p-1)!}.$$

Adjusting the summation index in the latter sum so that it starts at $p=0$ upon noting that $(a)_{p+1} = a(a+1)_p$ it is readily seen that

$$ {}_3F_2\left(\begin{matrix}f, a, c+1; \\ b, c;\end{matrix}\ 1\right) = {}_2F_1\left(\begin{matrix}f, a, \\ b;\end{matrix}\ 1\right) + \frac{af}{bc}{}_2F_1\left(\begin{matrix}f+1, a+1; \\ b+1;\end{matrix}\ 1\right).$$

If $\mathrm{Re}(b-a-f)>1$ we may apply Gauss's summation formula to each ${}_2F_1(1)$ thus obtaining

$$ \begin{aligned} {}_3F_2\left(\begin{matrix}f, a, c+1; \\ b, c;\end{matrix}\ 1\right) &= \frac{\Gamma(b)\Gamma(b-a-f)}{\Gamma(b-a)\Gamma(b-f)} + \frac{af}{bc}\frac{\Gamma(b+1)\Gamma(b-a-f-1)}{\Gamma(b-a)\Gamma(b-f)} \\ &= \frac{\Gamma(b)\Gamma(b-a-f-1)}{\Gamma(b-a)\Gamma(b-f)}\left[(b-a-f-1)+\frac{af}{c}\right]. \end{aligned}$$

It is easily verified that the expression in brackets is equal to $(c-a)(\alpha-f)/c$ thus giving equation (2.2).

In equation (2.2) now set $f=-n$, where n is a nonnegative integer. Hence

$$ {}_3F_2\left(\begin{matrix}-n, a, c+1; \\ b, c;\end{matrix}\ 1\right) = \frac{(c-a)(\alpha+n)}{c}\frac{\Gamma(b)\Gamma(b-a-1+n)}{\Gamma(b-a)\Gamma(b+n)}, \tag{2.3}$$

where from equation (1.1b) $\alpha = c(b-a-1)/(c-a)$. The condition $\mathrm{Re}(b-a+n) > 1$ becomes superfluous since the latter sum ${}_3F_2(1)$ terminates.

Since $\alpha + n = \alpha(1+\alpha)_n/(\alpha)_n$, equation (2.3) may be written as

$$ {}_3F_2\left(\begin{matrix}-n, a, c+1;\\ b, c;\end{matrix}\ 1\right) = \left[\frac{c-a}{c}\frac{c(b-a-1)}{c-a}\frac{\Gamma(b-a-1)}{\Gamma(b-a)}\right]\frac{(1+\alpha)_n(b-a-1)_n}{(\alpha)_n(b)_n}. $$

The expression in brackets reduces to unity thus proving the result given by equation (2.1).

Alternatively, the summation formula (2.1) may also be obtained by utilizing a two-term transformation for Clausen's series ${}_3F_2(1)$, which evidently is due originally to Kummer (1836) (see [1, p 142]) and states that

$$ {}_3F_2\left(\begin{matrix}a, b, c;\\ d, e;\end{matrix}\ 1\right) = \frac{\Gamma(e)\Gamma(d+e-a-b-c)}{\Gamma(e-c)\Gamma(d+e-a-b)}{}_3F_2\left(\begin{matrix}c, d-a, d-b;\\ d, d+e-a-b;\end{matrix}\ 1\right). \tag{2.4} $$

This result is additionally one of numerous two-term relations deduced by Whipple in 1925 and may easily be retrieved from the tables in Bailey's tract [2, section 3.5] by observing that $F_p(0; 1, 4) = F_p(0; 4, 5)$ (cf also [5, pp 104–7] and in particular [5, p 105, equation (16)]).

Since the transformation given by equation (2.4) is essentially recorded in tabular form in [2, 5, 9] it is not too well known. In fact, it was rederived in 1999 by Andrews *et al* [1, p 142] and by Exton [4] who obtained it by using elementary series manipulation and the summation theorems of Gauss and Saalschütz. In 2004, Rathie *et al* [8] rederived it again by essentially representing Clausen's series ${}_3F_2(1)$ by a beta integral transform of Gauss's function ${}_2F_1(z)$ followed by employing Euler's identity for the latter function to obtain a second beta transform representation for ${}_3F_2(1)$. (However, the authors in [8] neglected to mention that the condition $\mathrm{Re}\, c > 0$ which is necessary for the two respective beta transforms to exist can be waived by appealing to the principle of analytic continuation.)

In order to obtain equation (2.1) from equation (2.4) let $c = -n$, $b = d+1$ in the latter so that

$$ \begin{aligned} {}_3F_2\left(\begin{matrix}-n, a, d+1;\\ e, d;\end{matrix}\ 1\right) &= \frac{\Gamma(e)\Gamma(e-a-1+n)}{\Gamma(e+n)\Gamma(e-a-1)}{}_3F_2\left(\begin{matrix}-n, d-a, -1;\\ d, e-a-1;\end{matrix}\ 1\right) \\ &= \frac{(e-a-1)_n}{(e)_n}\left[1 + \frac{(-n)(d-a)(-1)}{d(e-a-1)}\right] \\ &= \frac{(e-a-1)_n}{(e)_n}\frac{d-a}{d(e-a-1)}\left[\frac{d(e-a-1)}{d-a} + n\right]. \end{aligned} $$

Now let $d \mapsto c$ and $e \mapsto b$. Thus since $\alpha = c(b-a-1)/(c-a)$ we have

$$ {}_3F_2\left(\begin{matrix}-n, a, c+1;\\ b, c;\end{matrix}\ 1\right) = \frac{(b-a-1)_n}{(b)_n}\frac{1}{\alpha}(\alpha+n). $$

Finally, recalling that $\alpha + n = \alpha(\alpha+1)_n/(\alpha)_n$ we deduce equation (2.1).

3. Reduction and transformation formulae

Let (H_h) denote the sequence of parameters $(H_1, H_2, \ldots, H_h)$ and for nonnegative integers n define the product of Pochhammer symbols $((H_h))_n \equiv (H_1)_n(H_2)_n \ldots (H_h)_n$, where when $h = 0$ the product is understood to reduce to unity. Consider now the double sum S defined by

$$ S \equiv \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{((H_h))_{m+n}((A_a))_m((B_b))_n}{((G_g))_{m+n}((C_c))_m((D_d))_n}\frac{x^m}{m!}\frac{y^n}{n!}, \tag{3.1} $$

which we assume to be absolutely convergent.

Employing series rearrangement equation (3.1) may be written as

$$S=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{((H_h))_n((A_a))_m((B_b))_{n-m}}{((G_g))_n((C_c))_m((D_d))_{n-m}}\frac{x^m}{m!}\frac{y^{n-m}}{(n-m)!}.$$

Since $(\alpha)_{n-m}=(-1)^m(\alpha)_n/(1-\alpha-n)_m$ (cf [9, p 239, equation (I.9)]) which provides when $\alpha=1$ in particular $1/(1)_{n-m}=1/(n-m)!=(-1)^m(-n)_m/n!$, we have

$$S=\sum_{n=0}^{\infty}\frac{((H_h))_n((B_b))_n}{((G_g))_n((D_d))_n}\frac{y^n}{n!}\sum_{m=0}^{n}\frac{(-n)_m((A_a))_m((1-D_d-n))_m}{((C_c))_m((1-B_b-n))_m}(-1)^{m(b-d+1)}\frac{(x/y)^m}{m!}$$

and this may be written simply as

$$S=\sum_{n=0}^{\infty}\frac{((H_h))_n((B_b))_n}{((G_g))_n((D_d))_n}\frac{y^n}{n!}{}_{a+d+1}F_{b+c}\left(\begin{matrix}-n,(A_a),(1-D_d-n);\\(C_c),(1-B_b-n);\end{matrix}\;(-1)^{b-d+1}\frac{x}{y}\right).$$

The double series defining S in equation (3.1) may be identified with a Kampé de Fériet function (see [11, p 63]) and so the latter result becomes

$$F^{h:a;b}_{g:c;d}\left[\begin{matrix}(H):(A);(B);\\(G):(C);(D);\end{matrix}\;x,y\right]=\sum_{n=0}^{\infty}\frac{((H))_n((B))_n}{((G))_n((D))_n}\frac{y^n}{n!}{}_{a+d+1}F_{b+c}\left(\begin{matrix}-n,(A),(1-D-n);\\(C),(1-B-n);\end{matrix}\;(-1)^{b-d+1}\frac{x}{y}\right),\tag{3.2}$$

where for brevity we have written $(H_h)=(H),(G_g)=(G)$, etc. (See [11, p 145, equation (30)] for the version of this result due to H M Srivastava.)

In particular, if $(B)=(D)$ (or equivalently $b=d=0$) and $x=-y$, then letting $(C)=(A')$ and $c=a'$, equation (3.2) reduces substantially to

$$F^{h:a;0}_{g:a';0}\left[\begin{matrix}(H)&:&(A)&;&—&;\\(G)&:&(A')&;&—&;\end{matrix}\;-y,y\right]=\sum_{n=0}^{\infty}\frac{((H))_n}{((G))_n}{}_{a+1}F_{a'}\left(\begin{matrix}-n,(A);\\(A');\end{matrix}\;1\right)\frac{y^n}{n!}.$$

Now letting $a=a'=2$, $(A)\mapsto(a,c+1),(A')\mapsto(b,c)$ and setting $y=x$ by using equation (2.1) we obtain the reduction formula

$$F^{h:2;0}_{g:2;0}\left[\begin{matrix}(H)&:&a,c+1&;&—&;\\(G)&:&b,c&;&—&;\end{matrix}\;-x,x\right]={}_{h+2}F_{g+2}\left(\begin{matrix}(H),b-a-1,\alpha+1;\\(G),b,\alpha;\end{matrix}\;x\right),\tag{3.3}$$

where α is given by equation (1.1b), i.e. $\alpha=c(1+a-b)/(a-c)$.

Exton obtained the garbled specialization [3, equation (8)] of equation (3.3) when $c=\frac{1}{2}a$ in which case $\alpha=1+a-b$. (See [6, equation (1)] for the corrected version of this result.) Another specialization already recorded in [10, p 31, equation (45)] is obtained from equation (3.3) by setting $b=c+1$ (in which case $\alpha=c$ so that $b=\alpha+1$) thus giving the reduction formula of lower order

$$F^{h:1;0}_{g:1;0}\left[\begin{matrix}(H)&:&a;&—;\\(G)&:&c;&—;\end{matrix}\;-x,x\right]={}_{h+1}F_{g+1}\left(\begin{matrix}(H),&c-a;\\(G),&c;\end{matrix}\;x\right).\tag{3.4}$$

Finally, letting $(H)=(G)$ (or equivalently $h=g=0$) in equation (3.3) we obtain the transformation formula

$$\mathrm{e}^x{}_2F_2\left(\begin{matrix}a,c+1;\\b,c;\end{matrix}\;-x\right)={}_2F_2\left(\begin{matrix}b-a-1,\alpha+1;\\b,\alpha;\end{matrix}\;x\right),$$

which gives equation (1.1a) when x is replaced by $-x$. (Note also that equation (3.4) reduces to a modified form of equation (1.2) when $(H)=(G)$ or $h=g=0$.)

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